# Matrix Decomposition-Based Adaptive Control of Noncanonical Form MIMO DT Nonlinear Systems

Yanjun Zhang, *Member, IEEE*, Ji-Feng Zhang, *Fellow, IEEE*, and Xiao-Kang Liu, *Member, IEEE*

***Abstract*—This article presents a new study on adaptive control of multi-input and multi-output (MIMO) discrete-time nonlinear systems with a noncanonical form involving parametric uncertainties. The adaptive control scheme employs a vector relative degree formulation to reconstruct the noncanonical system dynamics and derives a normal form. Then, a new matrix decomposition-based adaptive control scheme is proposed for the controlled plant with a vector relative degree $[1, 1, \ldots, 1]$ under some relaxed design conditions. In particular, the matrix decomposition technique is adopted to overcome the singularity problem during the adaptive estimation of an uncertain high-frequency gain matrix. The adaptive control scheme ensures closed-loop stability and asymptotic output tracking. An extension to the adaptive control of general canonical-form MIMO discrete-time nonlinear systems is also presented. Finally, through simulations, the effectiveness of the proposed control scheme is verified.**

***Index Terms*—Adaptive control, matrix decomposition, noncanonical form, output tracking.**

## I. Introduction

Adaptively controlling uncertain multi-input and multi-output (MIMO) nonlinear systems suffers from the singularity problem during the adaptive control design process. This problem is generally caused by adaptively estimating the uncertain high-frequency gain matrices. Estimating the latter matrices may be singular in parameter adaptation, leading to the singularity problems of the adaptive control laws. To solve the singularity problem, Morse introduced the matrix decomposition technique [1].

To date, several works addressed the singularity problems corresponding to the high-frequency gain matrices. For example, the work in [2] systematically summarized some matrix decomposition techniques commonly used in adaptive control of MIMO linear time-invariant systems considering continuous-time (CT) and discrete-time (DT), while the work in [3]–[5] studied matrix decomposition-based adaptive control problems for CT nonlinear systems. Matrix decomposition-based adaptive fuzzy/neural network control methods of CT nonlinear systems were also proposed [8]–[13]. Specifically, for CT and DT LTI systems, existing methods generally employed the transfer function matrices for the control designs, with the stability analysis relying on the small-gain lemma [2]. For CT nonlinear systems, popular methods commonly used robust control and approximation techniques; however, they usually needed the bound information of the eigenvalues corresponding to the high-frequency gain matrices [3]–[7]. Recently, the work in [8] developed a matrix decomposition-based adaptive control scheme for MIMO CT nonlinear systems under a linearly parameterized control framework.

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Yanjun Zhang is with the School of Automation, Beijing Institute of Technology, Beijing 100081, China (e-mail: yanjun@bit.edu.cn).

Ji-Feng Zhang is with the Key Laboratory of Systems and Control, Institute of Systems Science, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, and also with the School of Mathematics Sciences, University of Chinese Academy of Sciences, Bejing 100149, China (e-mail: jif@iss.ac.cn).

Xiao-Kang Liu is with the School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan 430074, China (e-mail: xiaokangliu@hust.edu.cn).

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However, applying the matrix decomposition technique to adaptive control designs of MIMO DT nonlinear systems is rare, especially for those in noncanonical forms. The noncanonical form refers to the system dynamics not being in the strict-feedback form and the output being generally a combination of some or all of the state variables. Accordingly, the canonical form means that the system dynamics are of the strict-feedback form, and the output is the first state variable. The canonical-form systems have explicit relative degrees that are crucial for adaptive control designs. The existing adaptive control methods of CT and DT nonlinear systems primarily focus on the canonical-form systems, which are not effective in controlling the noncanonical form systems. This is because the noncanonical form systems do not meet the canonical-form matching conditions [14]. On the other hand, there are no transfer functions or small-gain lemmas for DT nonlinear systems. Thus, the existing methods for LTI systems cannot be directly used. The control methods of [12] referring to the noncanonical form CT systems are also not applicable to control the noncanonical form MIMO DT systems due to the essential differences between the stability characterizations of the CT and DT systems.

Therefore, the singularity problem is still an open research case. To this end, we develop a matrix decomposition-based adaptive state feedback control scheme for a class of MIMO DT nonlinear systems. It solves the adaptive controller's singularity problem, and ensures the desired system performance. We also demonstrate that the proposed control scheme is applicable to control a general class of uncertain canonical-form nonlinear systems. Finally, through an example, we verify the validity of the proposed control scheme.

## II. Problem Statement

This section presents the system model and the problems to be addressed in this article.

### A. System Model

Consider the following MIMO DT nonlinear system:

$$x(t+1) = \Theta_f^* \phi_f(x(t)) + Bu(t),\ y(t) = Cx(t) \tag{1}$$

where $t \in \{0, 1, 2, \ldots\}$, $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n$, $y(t) = [y_1(t), y_2(t), \ldots, y_M(t)]^T \in \mathbb{R}^M$, and $u(t) = [u_1(t), u_2(t), \ldots, u_M(t)]^T \in \mathbb{R}^M$ are the state vector, the output vector, and the input vector, respectively, and $\Theta_f^*$, $B$, and $C$ are constant matrices defined as $B = [B_1, B_2, \ldots, B_M] \in \mathbb{R}^{n\times M}$ with $B_j = [b_{j1}, \ldots, b_{jn}]^T \in \mathbb{R}^n$, $C = [C_1^T, C_2^T, \ldots, C_M^T]^T \in \mathbb{R}^{M\times n}$ with $C_j^T = [c_{j1}, \ldots, c_{jn}]^T \in \mathbb{R}^n$,

$$\Theta_f^* = \begin{bmatrix} \theta_{f_1}^{*T} & & \\ & \ddots & \\ & & \theta_{f_n}^{*T} \end{bmatrix} \in \mathbb{R}^{n\times \sum_{i=1}^n p_i}$$

with $\theta_{f_i}^* = [\theta_{i1}^*, \theta_{i2}^*, \ldots, \theta_{ip_i}^*]^T \in \mathbb{R}^{p_i}$, and $\phi_f(x(t)) = [\phi_{f_1}^T(x(t)), \phi_{f_2}^T(x(t)), \ldots, \phi_{f_n}^T(x(t))]^T \in \mathbb{R}^{\sum_{i=1}^n p_i}$ with $\phi_{f_i}(x(t)) = [f_{i1}(x(t)), \ldots, f_{ip_i}(x(t))]^T \in \mathbb{R}^{p_i}$, $i = 1, \ldots, n$ and $f_{ik} : \mathbb{R}^n \to \mathbb{R}$, $k = 1, \ldots, p_i$, being smooth mappings. Note that the remaining elements of $\Theta_f^*$ not explicitly defined are set to zero. In this article, we assume that the mappings $f_{ik}$ are known and Lipschitz on $\mathbb{R}^n$, the matrices $\Theta_f^*, B$, and $C$ are unknown, and the state variables are measurable.

System (1) is of a noncanonical form with linearly parameterized uncertainties. Note that the results of this article can be extended to a general class of noncanonical form DT nonlinear systems of the form $x(t+1) = f(x(t)) + g(x(t))u(t), y_j(t) = C_j x(t)$, $k = 1, 2, \ldots, M$, where $g(x(t)) = [g_1(x(t)), g_2(x(t)), \ldots, g_n(x(t))]^T \in \mathbb{R}^{n\times M}$ such that $g_i : \mathbb{R}^n \to \mathbb{R}^M$ are nonlinear smooth mappings with linearly parameterized uncertainties. Given that the control design for the general system case involves a large amount of notation, to improve readability and preserve the article in a reasonable length, we use model (1) to show the control design details.

The control objective is to develop an adaptive state feedback control law for system (1), ensuring that all closed-loop signals are bounded and $y(t)$ tracks any given bounded reference output $y^*(t)$ asymptotically: $\lim_{t\to\infty}(y(t) - y^*(t)) = 0$.

### B. Research Problems

*Relative degrees:* Note that system (1) is of a noncanonical form, which is not suitable for adaptive control designs, and thus, we first reconstruct the system dynamics. In this article, we employ a relative degree-based reconstruction method to solve this problem. Spurred by the single-input and single-output (SISO) DT nonlinear systems where the relative degree concept has been systematically studied [15], in this article, we extend the relative degree concept to facilitate the MIMO DT nonlinear system (1).

Define $F(x, u) = \Theta_f^* \phi_f(x(t)) + Bu(t)$ and $F_0(x) = F(x, 0)$. Let notation $\circ$ denote a composition operation, i.e., $p_1 \circ p_2$ denotes that $p_1$ is a function of $p_2$ for any functions $p_1$ and $p_2$ with appropriate dimensions. Specifically, for any positive integer $k$, we define $F_0^k \circ F(x, u) = F_0(F_0^{k-1} \circ F(x, u))$ with $F_0^0 \circ F(x, u) = F(x, u)$ and $F_0 \circ F(x, u) = F(F(x, u), 0)$. Then, we define a matrix

$$G = \left[ C_1 \frac{\partial F_0^{\rho_1 - 1} \circ F(x, u)}{\partial u}, \ldots, C_M \frac{\partial F_0^{\rho_M - 1} \circ F(x, u)}{\partial u} \right] \tag{2}$$

which will be used to define the vector relative degrees.

The relative degree definitions proposed in [15] and [16] motivate us to develop the following definition.

*Definition 1:* System (1) has a global vector relative degree $[\rho_1, \rho_2, \ldots, \rho_M]$ ($\rho_j \geq 1$ and $\sum_{j=1}^M \rho_j \leq n$), if $C_j \frac{\partial F_0^{k_j} \circ F(x,u)}{\partial u} = 0$, $k_j = 0, 1, \ldots, \rho_j - 2$, $j = 1, 2, \ldots, M$; and $G$ in (2) is nonsingular for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^M$.

This definition specifies a general vector relative degree condition for system (1) and can be extended to define vector relative degrees for a general class of nonlinear systems.

*System dynamics reconstruction:* The following lemma specifies a relative degree dependent normal form.

*Lemma 1:* If system (1) has a vector relative degree $[\rho_1, \rho_2, \ldots, \rho_M]$ for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^M$, via a diffeomorphism $T(x(t)) = [\xi^T(t), \eta^T(t)]^T$ for $\xi(t) = [\xi_1^T(t), \ldots, \xi_M^T(t)]^T \in \mathbb{R}^{\sum_{j=1}^M \rho_j}$ with $\xi_j(t) = [\xi_{j1}, \ldots, \xi_{j\rho_j}]^T \in \mathbb{R}^{\rho_j}$ and $\eta(t) \in \mathbb{R}^{n - \sum_{j=1}^M \rho_j}$, then system (1) can be transformed into two subsystems: the output dynamics

$$\xi_{ji}(t+1) = \xi_{j(i+1)}(t),\ i = 1, \ldots, \rho_j - 1$$

$$\xi_{j\rho_j}(t+1) = C_j F_0^{\rho_j - 1} \circ F(x(t), u(t))$$

with $\xi_{j1}(t) = y_j(t)$, $j = 1, 2, \ldots, M$, such that $G$ in (2) is nonsingular for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^M$, and the internal dynamics

$$\eta(t+1) = q(\xi(t), \eta(t), u(t)) \tag{3}$$

where $q : \mathbb{R}^{\sum_{j=1}^M \rho_j} \times \mathbb{R}^{n - \sum_{j=1}^M \rho_j} \times \mathbb{R}^M \to \mathbb{R}^{n - \sum_{j=1}^M \rho_j}$ is a nonlinear smooth mapping.

To prove Lemma 1, we specify $\xi(t)$ and $\eta(t)$ with $\xi_{j1}(t) = y_j(t)$. Based on the Frobenius Theorem [16, Th. 1.4.1], one can find $n - \sum_{j=1}^M \rho_j$ vectors to construct $\eta(t)$. The proof of this lemma is similar to the case of CT nonlinear systems. For further details regarding the proof of the latter, the reader is referred to [16].

*Specification of the control problem:* If system (1) has a vector relative degree [1,1,...,1] for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^M$, then the output dynamics are

$$y(t+1) = \Theta_{cf}^* \phi_f(x(t)) + \Theta_{cb}^* u(t) \tag{4}$$

where $\Theta_{cb}^*$ is nonsingular, and $\Theta_{cf}^*$ and $\Theta_{cb}^*$ are in the forms $\Theta_{cf}^* = [\theta_{c_1 f}^*, \ldots, \theta_{c_M f}^*]^T$ and $\Theta_{cb}^* = [\theta_{c_1 b}^*, \ldots, \theta_{c_M b}^*]^T$ with $\theta_{c_j f}^* = C_j \Theta_f^*$ and $\theta_{c_j b}^* = [C_j B_1, C_j B_2, \ldots, C_j B_M]^T$.

A commonly used adaptive control law for (4) is in the form $u(t) = \Theta_{cb}^{-1}(t)(-\Theta_{cf}(t)\phi_f(x(t)) + v(t))$, where $\Theta_{cf}(t)$ and $\Theta_{cb}(t)$ are estimates of $\Theta_{cf}^*$ and $\Theta_{cb}^*$, respectively, and $v(t)$ is a designed signal. Note that $\Theta_{cb}(t)$ may be singular in the process of parameter adaptation, leading to the adaptive control law singularity. This article solves this problem by developing a matrix decomposition-based adaptive control scheme under a linearly parametric framework.

*Remark 1:* If system (1) has any higher order vector relative degree, the situation is quite different. For example, if system (1) has a vector relative degree [2,2,...,2], then the output dynamics are $y(t+2) = \Theta_{cf}^* \phi_f(\Theta_f^* \phi_f(x(t)) + Bu(t))$ such that $\Theta_{cf}^* \frac{\partial \phi_f(\Theta_f^* \phi_f(x) + Bu)}{\partial u}$ is nonsingular. Since $\phi_f$ is a nonlinear mapping process, $y(t+2)$ linearly depends on $\Theta_{cf}^*$ but nonlinearly depends on $\Theta_f^*, B$ and $u(t)$. Such characterizations impose two difficulties: how to simultaneously handle the linearly and nonlinearly parameterized uncertainties in $y(t+2)$ and how to derive an explicit adaptive control law. Although several works address adaptive control for systems with nonlinearly parameterized uncertainties and/or nonaffine control inputs [17]–[26], these methods cannot be directly employed to solve the problem discussed previously. Thus, this article mainly addresses the vector relative degree [1,1,...,1] case, while the high-order case shall be a future study.

## III. Adaptive Control Design

This section presents the control design details for system (1) with a vector relative degree [1,1,...,1], along with an extension of the adaptive control scheme.

### A. Design Conditions

For an adaptive control design, the control law $u(t)$ is designed of the basic form $u(t) = u(x(t), y^*(t), \Theta(t)) = u(T^{-1}(\xi(t), \eta(t)), y^*(t), \Theta(t))$, where $y^*(t)$ and $\Theta(t)$ denote the given reference output and a set of parameter estimates, respectively. Thus, (3) can be expressed as $\eta(t+1) = q(\xi(t), \eta(t), u(T^{-1}(\xi(t), \eta(t)), y^*(t), \Theta(t))) = Q(\xi(t), \eta(t), v(t))$, where $Q : \mathbb{R}^{\sum_{j=1}^{M} \rho_j} \times \mathbb{R}^{n-\sum_{j=1}^{M} \rho_j} \times \mathbb{R}^r \to \mathbb{R}^{n-\sum_{j=1}^{M} \rho_j}$ is a smooth mapping process, and $v(t) \in \mathbb{R}^r$ is a bounded signal depending on $y^*(t)$ and $\Theta(t)$. Based on this manipulation, we make the following assumption.

*Assumption 1:* The origin of system $\eta(t+1) = Q(0, \eta(t), 0)$ is globally exponentially stable, and $Q(\xi, \eta, v)$ is globally Lipschitz in $\xi$ and $v$.

In the literature, Assumption 1 is often called the input-to-state stable (ISS) condition [27], [28]. With Assumption 1, one can verify that if $\xi(t)$ and $v(t)$ are bounded, $\eta(t)$ is bounded.

When system (1) has a vector relative degree [1,1,...,1], the output dynamics are given by (4). Nevertheless, to utilize (4), we need the following assumption to decompose $\Theta_{cb}^*$.

*Assumption 2:* All leading principal minors of $\Theta_{cb}^*$, defined as $\Delta_i,\ i = 1, 2, \ldots, M$, are nonzero and their signs are known.

*Remark 2:* Based on Assumption 2, $\Theta_{cb}^*$ can be uniquely decomposed as $\Theta_{cb}^* = LD^*U$ for some unit lower triangular matrix $L$, some unit upper triangular matrix $U$, and

$$D^* = \operatorname{diag}\{d_1^*, d_2^*, \ldots, d_M^*\} = \operatorname{diag}\left\{\Delta_1, \frac{\Delta_2}{\Delta_1}, \ldots, \frac{\Delta_M}{\Delta_{M-1}}\right\}. \quad (5)$$

Furthermore, we can derive the SDU decomposition as $\Theta_{cb}^* = S^* D_s U_s$, where $S^* = LD^* D_s^{-1} L^T$ is a positive definite matrix, $U_s = D_s^{-1} L^{-1^T} D_s U$ is a unit upper triangular matrix, and

$$\begin{aligned} D_s &= \operatorname{diag}\{s_1^*, s_2^*, \ldots, s_M^*\} \\ &= \operatorname{diag}\{\operatorname{sign}[d_1^*]\gamma_1, \operatorname{sign}[d_2^*]\gamma_2, \ldots, \operatorname{sign}[d_M^*]\gamma_M\} \end{aligned} \quad (6)$$

such that $\gamma_i,\ i = 1, \ldots, M$, are chosen positive constants. □

### B. Parameterized Model and Adaptive Control Law

From Assumption 2, a parameterized model of (4) is derived as

$$S^{*-1} y(t+1) = D_s \Theta_1^* \phi_f(x(t)) + D_s \Theta_2^* u(t) + D_s u(t) \quad (7)$$

where $\Theta_1^* = D_s^{-1} S^{*-1} \Theta_{cf}^*$, and $\Theta_2^* = U_s - I$ of the form

$$\Theta_2^* = \begin{bmatrix} 0 & \theta_{212}^* & \theta_{213}^* & \cdots & \theta_{21\,M}^* \\ 0 & 0 & \theta_{223}^* & \cdots & \theta_{22\,M}^* \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \theta_{2(M-1)M}^* \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

To ensure the desired system performance, the adaptive control law is designed as

$$\begin{aligned} u(t) = &-\Theta_2(t)u(t) - \Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t+1) \\ &- \Theta_3(t)A_m(y(t) - y^*(t)) \end{aligned} \quad (8)$$

where $\Theta_i(t),\ i = 1, 2, 3$, are the estimates of $\Theta_1^*$, $\Theta_2^*$, and $(S^* D_s)^{-1}$, respectively, $A_m$ is a stable matrix, and $y^*(t) \in R^M$ is a given bounded reference output signal. In particular, $\Theta_2(t)$ has the form

$$\Theta_2(t) = \begin{bmatrix} 0 & \theta_{212}(t) & \theta_{213}(t) & \cdots & \theta_{21\,M}(t) \\ 0 & 0 & \theta_{223}(t) & \cdots & \theta_{22\,M}(t) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \theta_{2(M-1)M}(t) \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (9)$$

where $\theta_{2ij}(t)$ are estimates of $\theta_{2ij}^*$.

*Remark 3:* The adaptive control law $u(t)$ can be calculated as follows. Let $w(t) = [w_1(t), w_2(t), \ldots, w_M(t)]^T = -\Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t+1) - \Theta_3(t)A_m(y(t) - y^*(t))$. Note that $w(t)$ is independent of $u(t)$, and $u(t)$ can be expressed as $u(t) = -\Theta_2(t)u(t) + w(t)$. Then, $u(t)$ can be sequentially calculated as

$$\begin{aligned} u_M(t) &= w_M(t) \\ u_{M-1}(t) &= -\theta_{2(M-1)M}(t)u_M(t) + w_{M-1}(t) \\ &\vdots \\ u_1(t) &= -\theta_{211}(t)u_2(t) - \cdots - \theta_{21\,M}(t)u_M(t) + w_1(t). \end{aligned} \quad (10)$$

### C. Tracking Error Model

We define the tracking error $e(t) = y(t) - y^*(t) \in \mathbb{R}^M$. Substituting (8) in (7) yields $S^{*-1}y(t+1) = -D_s\tilde{\Theta}_1(t)\phi_f(x(t)) - D_s\tilde{\Theta}_2(t)u(t) + D_s\Theta_3(t)y^*(t+1) - D_s\Theta_3(t)A_m e(t)$, where $\tilde{\Theta}_i(t) = \Theta_i(t) - \Theta_i^*(t),\ i = 1, 2.$ Adding $-S^{*-1}y^*(t+1) + S^{*-1}A_m e(t)$ to the above equation, we obtain

$$\begin{aligned} &S^{*-1}(e(t+1) + A_m e(t)) \\ &= -D_s(\tilde{\Theta}_1(t)\phi_f(x(t)) + \tilde{\Theta}_2(t)u(t) \\ &+ \tilde{\Theta}_3(t)(A_m e(t) - y^*(t+1))) \end{aligned} \quad (11)$$

where $\tilde{\Theta}_3(t) = \Theta_3(t) - \Theta_3^*(t)$. We define $P_m(z) = zI + A_m$, $\varphi(t) = [-\phi_f^T(x(t)), -u^T(t), -(A_m e(t) - y^*(t+1))^T]^T$,

$$\Psi(t) = [\Theta_1(t), \Theta_2(t), \Theta_3(t)] \quad (12)$$

and $\tilde{\Psi}(t) = \Psi(t) - \Psi^*(t) = [\tilde{\Theta}_1(t), \tilde{\Theta}_2(t), \tilde{\Theta}_3(t)]$, where $z$ denotes the time advance operator. Then, (11) is rewritten as

$$P_m(z)[e](t) = S^* D_s \tilde{\Psi}(t)\varphi(t). \quad (13)$$

To implement $u$ from (10), we express $\tilde{\Psi}(t)\varphi(t)$ as

$$\tilde{\Psi}(t)\varphi(t) = [(\Psi_1(t) - \Psi_1^*)\varphi_1(t), \ldots, (\Psi_M(t) - \Psi_M^*)\varphi_M(t)]^T$$

where

$$\begin{aligned} \Psi_1(t) - \Psi_1^* &= [\theta_{212}(t) - \theta_{212}^*, \ldots, \theta_{21\,M}(t) - \theta_{21\,M}^* \\ &\qquad \tilde{\Theta}_{11}(t), \tilde{\Theta}_{31}(t)] \\ &\vdots \\ \Psi_{M-1}(t) - \Psi_{M-1}^* &= [\theta_{2(M-1)M}(t) - \theta_{2(M-1)M}^* \\ &\qquad \tilde{\Theta}_{1(M-1)}(t), \tilde{\Theta}_{3(M-1)}(t)] \\ \Psi_M(t) - \Psi_M^* &= [\tilde{\Theta}_{1\,M}(t), \tilde{\Theta}_{2\,M}(t)] \end{aligned}$$

and

$$\varphi_1(t) = \left[-u_2(t), -u_3(t), \ldots, -u_M(t), -\phi_f^T(x(t)) \right.$$
$$\left. -(A_m e(t) - y^*(t+1))^T\right]^T \quad (14)$$
$$\vdots$$
$$\varphi_{M-1}(t) = \left[-u_M(t), -\phi_f^T(x(t)) \right.$$
$$\left. -(A_m e(t) - y^*(t+1))^T\right]^T \quad (15)$$
$$\varphi_M(t) = \left[-\phi_f^T(x(t)), -(A_m e(t) - y^*(t+1))^T\right]^T. \quad (16)$$

We also introduce $h_0(z) = z - \alpha$ and $h(z) = \frac{1}{h_0(z)}$, with $0 < \alpha < 1$. A filtered error is defined as

$$\bar{e}(t) = P_m(z)h(z)[e](t) = [\bar{e}_1(t), \ldots, \bar{e}_M(t)]^T. \quad (17)$$

Operating both sides of (13) by the stable filter $h(z)$, we obtain

$$\bar{e}(t) = S^* D_s h(z)[\tilde{\Psi}\varphi](t) \quad (18)$$

which is the expected tracking error model.

## D. Estimation Error

To design the update laws, we define an estimation error

$$\epsilon(t) = \bar{e}(t) + \Phi(t)\sigma(t) \quad (19)$$

where $\Phi(t)$ is the estimate of $S^* D_s$ and $\sigma(t) = [\sigma_1(t), \sigma_2(t), \ldots, \sigma_M(t)]^T$ with

$$\sigma_j(t) = \Psi_j(t)\delta_j(t) - h(z)[\Psi_j\varphi_j](t), \delta_j(t) = h(z)[\varphi_j](t). \quad (20)$$

Note that $\bar{e}(t), \epsilon(t), \sigma_j(t), \delta_j(t)$ are known at the current time instant. From (18)–(20), $\epsilon(t)$ can be expressed as

$$\epsilon(t) = S^* D_s[\tilde{\Psi}_1\delta_1(t), \ldots, \tilde{\Psi}_M\delta_M(t)]^T + \tilde{\Phi}(t)\sigma(t).$$

## E. Parameter Update Laws

To implement the control signal, we need to update $\Theta_i, i = 1, 2, 3$. From (12), updating $\Theta_i$ is equivalent to updating $\Psi(t)$. Note that $\epsilon(t)$ depends on $\Phi(t)$ (the estimate of $S^* D_s$); thus, we further need to update $\Phi(t)$. Using (19)–(20), the parameter update laws are designed as

$$\Psi_j^T(t+1) = \Psi_j^T(t) - \frac{\text{sign}\{d_j^*\}\gamma_j\epsilon_j(t)\delta_j(t)}{m^2(t)}, \ j = 1, 2, \ldots, M \quad (21)$$

$$\Phi(t+1) = \Phi(t) - \frac{\beta\epsilon(t)\sigma^T(t)}{m^2(t)} \quad (22)$$

where

$$m(t) = \sqrt{1 + \sum_{j=1}^M \sigma_j^2(t) + \sum_{j=1}^M \delta_j^T(t)\delta_j(t)} \quad (23)$$

$\text{sign}\{d_j^*\}$ and $\gamma_j$ are specified in (5) and (6) such that $0 < \gamma_j^2 < 2\lambda_{\min}\{S^{*-1}\}$, and $\beta \in \mathbb{R}$ is a constant parameter such that $0 < \beta < \frac{2\lambda_{\min}\{S^{*-1}\}}{\lambda_{\max}\{S^{*-1}\}}$ with $\lambda_{\min}\{S^{*-1}\}$ and $\lambda_{\max}\{S^{*-1}\}$ denoting the minimum and maximum eigenvalues of $S^{*-1}$.

Next, we derive the following lemma to specify some key properties of $\Psi_j(t)$ and $\Phi(t)$.

*Lemma 2:* The parameter update laws of (21) and (22) ensure that, for $j = 1, 2, \ldots, M$,

1) $\Psi_j(t) \in L^\infty$ and $\Phi_j(t) \in L^\infty$;
2) $\frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty$, $\Psi_j(t+1) - \Psi_j(t) \in L^2 \cap L^\infty$, and $\Phi(t+1) - \Phi(t) \in L^2 \cap L^\infty$;
3) $\lim_{t\to\infty} \frac{\epsilon(t)}{m(t)} = 0$, $\lim_{t\to\infty}(\Psi_j(t+1) - \Psi_j(t)) = 0$, and $\lim_{t\to\infty}(\Phi(t+1) - \Phi(t)) = 0$.

*Proof:* Consider a positive definite function of the form $V(\tilde{\Psi}_j, \tilde{\Phi}) = \sum_{j=1}^M \tilde{\Psi}_j\tilde{\Psi}_j^T + \frac{1}{\beta}\text{tr}[\tilde{\Phi}^T S^{*-1}\Phi]$. Then,

$$V(\tilde{\Psi}_j(t+1), \tilde{\Phi}(t+1)) - V(\tilde{\Psi}_j(t), \tilde{\Phi}(t))$$
$$= -\sum_{j=1}^M \left(\frac{2\text{sign}\{d_j^*\}\gamma_j\epsilon_j(t)\tilde{\Psi}_j(t)\delta_j(t)}{m^2(t)} - \frac{\gamma_j^2\epsilon_j^2(t)\delta_j^T(t)\delta_j(t)}{m^4(t)}\right)$$
$$-\text{tr}\left[\frac{2\tilde{\Phi}^T(t)S^{*-1}\epsilon(t)\sigma^T(t)}{m^2(t)} - \frac{\beta(\epsilon(t)\sigma^T(t))^T S^{*-1}\epsilon(t)\sigma^T(t)}{m^4(t)}\right]. \quad (24)$$

Using the properties of matrix trace: $\text{tr}[X_1] = \text{tr}[X_1^T]$ and $\text{tr}[X_2X_3] = \text{tr}[X_3X_2]$ of any matrices $X_i, i = 1, 2, 3$, of appropriate dimensions, we derive

$$\text{tr}\left[\frac{2\tilde{\Phi}^T(t)S^{*-1}\epsilon(t)\sigma^T(t)}{m^2(t)}\right] = \frac{2\epsilon^T(t)S^{*-1}\tilde{\Phi}(t)\sigma(t)}{m^2(t)} \quad (25)$$

$$\text{tr}\left[\frac{\beta(\epsilon(t)\sigma^T(t))^T S^{*-1}\epsilon(t)\sigma^T(t)}{m^4(t)}\right] = \frac{\beta\epsilon^T(t)S^{*-1}\epsilon(t)\sigma^T(t)\sigma(t)}{m^4(t)}. \quad (26)$$

Letting $\chi(t) = \text{diag}\{\gamma_1^2\delta_1^T(t)\delta_1(t), \ldots, \gamma_M\delta_M^T(t)\delta_M(t)\}$ and substituting (25)–(26) in (24), we have

$$V(\tilde{\Psi}_j(t+1), \tilde{\Phi}(t+1)) - V(\tilde{\Psi}_j(t), \tilde{\Phi}(t))$$
$$= \frac{\epsilon^T(t)\left(2S^{*-1} - \frac{\chi(t) + \beta S^{*-1}\sigma^T(t)\sigma(t)}{m^2(t)}\right)\epsilon(t)}{m^2(t)}.$$

Since $S^{*-1}$ is a positive definite matrix, an orthogonal matrix $T_s \in \mathbb{R}^{M\times M}$ exists such that $T_s^T T_s = I$ and $T_s^T S^{*-1} T_s = \text{diag}\{\lambda_1, \lambda_2 \ldots, \lambda_M\} = P_s$ with $\lambda_j$ being the eigenvalues of $S^{*-1}$. Then, from (23), we have

$$2S^{*-1} - \frac{\chi(t) + \beta S^{*-1}\sigma^T(t)\sigma(t)}{m^2(t)}$$
$$\geq T_s\left(2\lambda_{\min}[S^{*-1}]I - \max\{\max_j\{\gamma_j^2\}, \beta\lambda_{\max}[S^{*-1}]\}I\right)T_s^T$$
$$= \left(2\lambda_{\min}[S^{*-1}] - \max\{\max_j\{\gamma_j^2\}, \beta\lambda_{\max}[S^{*-1}]\}\right)I$$

which implies that $2S^{*-1} - \frac{\chi(t) + \beta S^{*-1}\sigma^T(t)\sigma(t)}{m^2(t)} \geq \alpha_0 I$, for some $\alpha_0 > 0$. Hence, we obtain $V(\tilde{\Psi}_j(t+1), \tilde{\Phi}(t+1)) - V(\tilde{\Psi}_j(t), \tilde{\Phi}(t)) \leq -\alpha_0\frac{\epsilon^2(t)}{m^2(t)}$, which implies that $V$ is nonincreasing and $\frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty$.

Thus, $\Psi_j(t) \in L^\infty$ and $\Phi(t) \in L^\infty$. From (21) and (22), the property $\frac{\epsilon(t)}{m(t)} \in L^2 \cap L^\infty$ yields $\Psi_j(t+1) - \Psi_j(t) \in L^2 \cap L^\infty$ and $\Phi(t+1) - \Phi(t) \in L^2 \cap L^\infty$. Due to $\frac{\epsilon(t)}{m(t)}$, $\Psi_j(t+1) - \Psi_j(t)$, and $\Phi(t+1) - \Phi(t)$ all belonging to $L^2$, the properties specified in 3) of Lemma 2 hold.

Lemma 2 shows some desired properties of the parameter estimates. Given the adaptive control law presented, next, we will analyze the closed-loop system performance.

## F. Stability Analysis

To proceed, some useful notation should be introduced: $c$ denotes a signal bound; $\tau(t)$ denotes a generic $L^2 \cap L^\infty$ function that goes to zero as $t \to \infty$; and $L^{\infty e}$ denotes the set $L^{\infty e} = \{r(t) | \forall s < \infty, r_s(t) \in L^{\infty}\}$ for any function $r(t)$ with $r_s(t) = r(t)$, $t \leq s$, and $r_s(t) = 0$, $t > s$.

We first give the following result that will be used for the stability analysis performed shortly.

*Lemma 3:* For DT signals $r_i(t) \in \mathbb{R}^{p_i}$, $i = 1, 2, 3$, with $p_1 = p_2$, such that $r_1(t) = h(z)[r_2](t)$ and $r_2(t), r_3(t) \in L^{\infty e}$, if $\|r_2(t)\| \leq \tau_1(t) \sup_{k \leq t} \|r_3(k)\| + \tau_2(t)\ \forall t \geq 0$, then $\|r_1(t)\| \leq \tau_3(t) \sup_{k \leq t} \|r_3(k)\| + \tau_4(t)\ \forall t \geq 0$, where $\tau_i(t)$ are all $L^2 \cap L^{\infty}$ functions.

*Proof:* Let $h_z(t)$ denote the impulse response function of $h(z)$. It can be verified that $h_z(t) = \frac{1}{\alpha}(\alpha^t - \delta(t)) \geq 0\ \forall t \geq 0$, where $\delta(t)$ denotes the unit impulse response. Then, with $r_1(t) = h(z)[r_2](t)$ and ignoring the exponentially decaying effect of the initial conditions, we obtain $r_1(t) = \sum_{k=0}^{t} h_z(t-k) r_2(k)$. Let $\varepsilon_3(t) = \sum_{k=0}^{t} h_z(t-k)\tau_1(k)$, $\varepsilon_4(t) = \sum_{k=0}^{t} h_z(t-k)\tau_2(k)$, which can also be expressed as $\varepsilon_3(t) = h(z)[\tau_1](t)$, $\varepsilon_4(t) = h(z)[\tau_2](t)$. Note that $h(z)$ is stable, and $\tau_i(t) \in L^2 \cap L^{\infty}$, $i = 1, 2$. Thus, we obtain $\varepsilon_j(t) \in L^2 \cap L^{\infty}$, $j = 3, 4$. Then, by setting $\tau_3(t) = \varepsilon_3(t)$ and $\tau_4(t) = \varepsilon_4(t)$, we complete the proof.

Next, we provide the main result of this article as follows.

*Theorem 1:* Under Assumptions 1 and 2, the adaptive control law (8) with the parameter update laws (21) and (22), applied to system (1) with a vector relative degree [1,1,...,1], ensures closed-loop stability and asymptotic output tracking: $\lim_{t\to\infty}(y(t) - y^*(t)) = 0$.

*Proof:* The proof contains four steps.

*Step 1: Show* $\sum_{j=1}^{M} |\sigma_j(t)| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau$. Based on the fact that $\|y^*(t)\| \leq c$ ($\tau$ and $c$ were defined in Lemma 3) and $f_{ik}(x(t))$ are globally Lipschitz in $x(t)$, we derive from (14)–(16) that $\|\varphi_j(t)\| \leq c\|\xi(t)\| + c\|\eta(t)\| + c\|e(t)\| + c$. Then, with Assumption 1 and $\xi(t) = y(t)$, we have

$$\|\varphi_j(t)\| \leq c\|e(t)\| + c. \tag{27}$$

Since $\frac{z}{h_0(z)}$ is proper and stable, $\|\frac{z}{h_0(z)}[\varphi_j](t)\| \leq c + c \sup_{k \leq t} \|e(k)\|$. From (20), we have

$$\begin{aligned} &h_0(z)[\sigma_j](t) = h_0(z)[\Psi_j \delta_j](t) - \Psi_j(t)\varphi_j(t) \\ &= \Psi_j(t+1)\frac{z}{z-\alpha}[\varphi_j](t) + \Psi_j(t)\frac{z-\alpha-z}{z-\alpha}[\varphi_j](t) - \Psi_j(t)\varphi_j(t) \\ &= (\Psi_j(t+1) - \Psi_j(t))\frac{z}{z-\alpha}[\varphi_j](t) \end{aligned}$$

which reveals that $\sigma_j(t)$ can be expressed as

$$\sigma_j(t) = h(z)\left[(z-1)[\Psi_j]\frac{z}{h_0(z)}[\varphi_j]\right](t). \tag{28}$$

With $|(z-1)[\Psi_j](t)\frac{z}{h_0(z)}[\varphi_j](t)| \leq \|\Psi_j(t+1) - \Psi_j(t)\| \cdot \|\frac{z}{h_0(z)}[\varphi_j](t)\|$, it follows from Lemma 2 that

$$|(z-1)[\Psi_j](t)\frac{z}{h_0(z)}[\varphi_j](t)| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau. \tag{29}$$

From (28), (29), and Lemma 3, we obtain $|\sigma_j(t)| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau$, $\forall j = 1, 2, \ldots, M$. Thus, it follows that

$$\sum_{j=1}^{M} |\sigma_j(t)| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau. \tag{30}$$

*Step 2: Show* $m(t) \leq c \sup_{k \leq t} \|e(k)\| + c$. From the definition of $\delta_j(t)$ in (20), and given that $h(z)$ is strictly proper and stable, we obtain $\|\delta_j(t)\| \leq c \sup_{k \leq t} \|\varphi_j(k)\| + c$. Then, from (27),

$$\|\delta_j(t)\| \leq c \sup_{k \leq t} \|e(k)\| + c. \tag{31}$$

From the definition of $m(t)$ in (23), we derive $m(t) \leq 1 + \sum_{j=1}^{M} |\sigma_j(t)| + \sum_{j=1}^{M} \|\delta_j(t)\|$. Then, together with (30) and (31), we have

$$m(t) \leq c \sup_{k \leq t} \|e(k)\| + c. \tag{32}$$

*Step 3: Show* $e(t) \in L^{\infty}$. From (19), we get $\|\bar{e}(t)\| \leq \|\epsilon(t)\| + \|\Phi(t)\sigma(t)\| \leq m(t)\|\frac{\epsilon(t)}{m(t)}\| + \|\Phi(t)\sigma(t)\|$. Since $\frac{\epsilon(t)}{m(t)} \in L^2 \cap L^{\infty}$, $\Phi(t) \in L^{\infty}$, combined with (30), we derive $\|\bar{e}(t)\| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau$. Then, we obtain

$$\sup_{k \leq t} \|\bar{e}(k)\| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau. \tag{33}$$

Let $P_m^{-1}(z)$ denote the inverse of $P_m(z)$. Then, $P_m^{-1}(z)h_0(z)$ is proper and stable. From (17) and (33), we derive that $\|e(t)\| \leq c \sup_{k \leq t} \|\bar{e}(k)\| + c \leq \tau \sup_{k \leq t} \|e(k)\| + c$ which implies $e(t) \in L^{\infty}$.

*Step 4:* Show closed-loop stability and $\lim_{t\to\infty}(y(t) - y^*(t)) = 0.$: Since $\sup_{k \leq t} \|\bar{e}(k)\| \leq \tau \sup_{k \leq t} \|e(k)\| + \tau$ and $e(t) \in L^{\infty}$, we have $\bar{e}(t) \in L^2 \cap L^{\infty}$. With $P_m^{-1}(z)h_0(z)$ being proper and stable, we obtain $e(t) \in L^2 \cap L^{\infty}$. Note that for any DT signal belonging to $L^2$, it certainly converges to zero. Thus, we conclude that $\lim_{t\to\infty} e(t) = 0$.

With the boundedness of $e(t)$, from (27), (30), (31), and (32), we obtain the boundedness of $\varphi_j(t), \sigma_j(t), \delta_j(t)$, and $m(t)$. Moreover, from $\xi(t) = y(t)$, we derive the boundedness of $\xi(t)$, which follows from Assumption 1 that $\eta(t) \in L^{\infty}$. Thus, based on $T(x) = [\xi^T, \eta^T]^T$, we have the boundedness of $x(t)$. From (8), we have the boundedness of $u(t)$. □

Next, an extension of the proposed adaptive control scheme is presented.

### G. Extension to Canonical-Form MIMO DT Systems

This part extends the application of the proposed control scheme to the adaptive control of a general class of canonical-form MIMO DT nonlinear systems.

*System model:* We consider the following canonical-form MIMO DT nonlinear system

$$\begin{aligned} y_i(t+\rho_i) &= f_i(x(t)) + \sum_{j=1}^{M} g_{ij}(x(t))u_j(t), i = 1, 2, \ldots, M \\ \eta(t+1) &= q(\xi(t), \eta(t), u(t)) \end{aligned} \tag{34}$$

where $y_i \in \mathbb{R}$, $\eta \in \mathbb{R}^{r_\eta}$, $u_i \in \mathbb{R}$, $f_i \in \mathbb{R}$, $g_{ij} \in \mathbb{R}$, $i, j = 1, 2, \ldots, M$, are globally Lipschitz functions with linearly parameterized uncertainties, and $q \in \mathbb{R}^{r_\eta}$ is a nonlinear smooth mapping. For system (34), the state vector is

$$\begin{aligned} x(t) = [&y_1(t), \ldots, y_1(t+\rho_1-1), \ldots, \\ &y_M(t), \ldots, y_M(t+\rho_M-1), \eta^T(t)]^T \in \mathbb{R}^L. \end{aligned}$$

The input vector and the output vector are $u(t) = [u_1(t), \ldots, u_M(t)]^T \in \mathbb{R}^M$ and $y(t) = [y_1(t), \ldots, y_M(t)]^T \in \mathbb{R}^M$, respectively, with $L = \sum_{i=1}^{M} \rho_i + r_\eta$. We assume that $x(t)$ is measurable, which implies that $y_i(t+k)$, $i = 1, 2, \ldots, M$, $k = 1, \ldots, \rho_i - 1$, can be used for the adaptive control law design.

*Assumption:* System (34) needs to satisfy the following conditions.

i) The internal dynamics are ISS, which is similar to Assumption 1;
ii) System (34) has a well-defined vector relative degree, that is, $\{g_{ij}(x)\}$ is nonsingular for all $x \in \mathbb{R}^n$.
iii) $\{g_{ij}(x)\}$ can be decomposed into the form $\Theta_g^*\Phi_g(x)$ with an unknown constant square matrix $\Theta_g^*$ and a known time-varying square matrix $\Phi_g(x)$ such that $\det\{\Phi_g(x)\}$ is away from zero.

*Parameterized model:* The output dynamics are expressed as $[y_1(t+\rho_1),\ldots,y_M(t+\rho_M)]^T = \Theta_{cf}^*\phi_f(x(t)) + S^*D_sU_s\bar{u}(t)$, where $\Theta_{cf}^*\phi_f(x(t))$ is a parameterized model of the vector $[f_1(x(t)),\ldots,f_M(x(t))]^T$, $S^*D_sU_s$ is a decomposition of $\Theta_g^*$, and $\bar{u}(t) = \Phi_g(x(t))u(t)$. Based on assumptions ii) and iii), we observe that $\Phi_g(x(t))$ is always nonsingular with a bounded inverse for all $x \in \mathbb{R}^n$. Then, we derive a parameterized model of the output dynamics as

$$S^{*-1}\begin{bmatrix} y_1(t+\rho_1) \\ \vdots \\ y_M(t+\rho_M) \end{bmatrix} = D_s\Theta_1^*\phi_f(x(t)) + D_s\Theta_2^*\bar{u}(t) + D_s\bar{u}(t) \quad (35)$$

where $\Theta_1^*$ and $\Theta_2^*$ have the same expressions as $\Theta_1^*$ and $\Theta_2^*$ of the vector relative degree [1,1,...,1] case.

*Adaptive control law:* Motivated by (8), with $y^*(t) = [y_1^*(t),\ldots,y_M^*(t)]^T$, the adaptive control law is designed as

$$\begin{aligned} u(t) &= \Phi_g^{-1}(x(t))\bar{u}(t) \\ \bar{u}(t) &= -\Theta_2(t)\bar{u}(t) - \Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t+\rho) \\ &\quad -\Theta_3(t)\begin{bmatrix} \sum_{i=0}^{\rho_1-1} a_{1i}(y_1(t+i) - y_1^*(t+i)) \\ \vdots \\ \sum_{i=0}^{\rho_M-1} a_{Mi}(y_M(t+i) - y_M^*(t+i)) \end{bmatrix} \end{aligned} \quad (36)$$

where $\Theta_i(t), i = 1,2,3,$ have the same expressions as $\Theta_i(t)$ of the vector relative degree [1,1,...,1] case. In particular, the parameters $a_{ji}$ are set as constants such that all zeros of $P_{mi}(z),\ i = 1,2,\ldots,M$, are inside the unit circle of the complex $z$-plane, where

$$P_{mi}(z) = z^{\rho_i} + a_{i(\rho_i-1)}z^{\rho_i-1} + \cdots + a_{i1}z + a_{i0}.$$

Note that $\Theta_2(t)$ in (36) has the same expression as that in (9). Thus, $\bar{u}(t)$ can also be calculated based on the procedure of Remark 3.

*Tracking error model:* Let $e(t) = y(t) - y^*(t)$. Substituting (36) in (35) provides

$$\begin{aligned} & S^{*-1}\begin{bmatrix} e_1(t+\rho_1) + \sum_{i=0}^{\rho_1-1} a_{1i}e_i(t+i) \\ \vdots \\ e_M(t+\rho_M) + \sum_{i=0}^{\rho_M-1} a_{Mi}e_M(t+i) \end{bmatrix} \\ &= -D_s\tilde{\Theta}_1(t)\phi_f(x(t)) - D_s\tilde{\Theta}_2(t)\bar{u}(t) \\ &\quad -D_s\tilde{\Theta}_3(t)\begin{bmatrix} \sum_{i=0}^{\rho_1-1} a_{1i}e_i(t+i) - y_1^*(t+\rho_1) \\ \vdots \\ \sum_{i=0}^{\rho_M-1} a_{Mi}e_M(t+i) - y_M^*(t+\rho_M) \end{bmatrix} \end{aligned} \quad (37)$$

where $\Theta_3(t)$ and $\tilde{\Theta}_3(t)$ have the same expressions as $\Theta_3(t)$ and $\tilde{\Theta}_3(t)$ of the vector relative degree [1,1,...,1] case. Let $\Psi(t) = [\Theta_1(t),\Theta_2(t),\Theta_3(t)]$, $\tilde{\Psi}(t) = \Psi(t) - \Psi^*(t) = [\tilde{\Theta}_1(t),\tilde{\Theta}_2(t),\tilde{\Theta}_3(t)]$, $P_m(z) = \mathrm{diag}\{P_{m1}(z),\ldots,P_{mM}(z)\}$, and

$$\begin{aligned} \varphi(t) = [&-\phi_f^T(x(t)), -\bar{u}^T(t), -\sum_{i=0}^{\rho_1-1} a_{1i}e_i(t+i) + y_1^*(t+\rho_1) \\ &\cdots, -\sum_{i=0}^{\rho_M-1} a_{Mi}e_M(t+i) + y_M^*(t+\rho_M)]^T. \end{aligned}$$

Then, (37) can be expressed as

$$P_m(z)[e](t) = S^*D_s\tilde{\Psi}(t)\varphi(t) \quad (38)$$

which is the expected tracking error model.

With the adaptive control law (36), the tracking error model (38), and following the design procedure of the vector relative degree [1,1,...,1] case, we sequentially derive the corresponding estimation errors and parameter update laws, and prove that the adaptive control law (36) can ensure closed-loop stability and asymptotic output tracking. Nevertheless, we omit the relevant details to optimize the reading flow.

We have developed a matrix decomposition-based adaptive control scheme for system (1) with vector relative degree [1,1,...,1]. Moreover, we demonstrated that the control scheme is applicable to adaptively control canonical-form MIMO DT systems with a general vector relative degree.

## IV. Simulation Study

This section demonstrates the design procedure and verifies the effectiveness of the proposed control method.

### A. System Model

Consider the following system model

$$\begin{aligned} x(t+1) &= f(x(t)) + B_1u_1(t) + B_2u_2(t) \\ y_j(t) &= C_jx(t),\ j = 1,2 \end{aligned} \quad (39)$$

where $x(t) = [x_1(t), x_2(t), x_3(t)] \in \mathbb{R}^3$ is the state vector, $u_j$ and $y_j$, $j = 1,2$, are the input and output variables, respectively, $B_1 = [1,0,1]^T$, $B_2 = [1,2,0]^T$, $C_1 = [-1,0,0]$, $C_2 = [1,-2,-1]$, and $f(x) = [f_1(x), f_2(x), f_3(x)]^T$. Moreover, $f_1 = \theta_{f_1}^{*T}\phi_{f_1} = [1.2, 2.08, 0.24][x_1, x_1\sin x_3, \sqrt{1+x_2^2}]^T$, $f_2 = \theta_{f_2}^{*T}\phi_{f_2} = [0.16, 0.4][x_2\sin x_1, x_1]^T$, and $f_3 = \theta_{f_3}^{*T}\phi_{f_3} = [0.48, 1.6][x_3, \sin(x_1)]^T$. In this simulation, we assume that $B_j, C_j, \theta_{f_i}^*$, $i = 1,2,3$, $j = 1,2$, are all unknown, and $\phi_{f_i}, i = 1,2,3$, are known.

### B. System Transformation

From the $B_j, C_j,\ j = 1,2$ parameters, we obtain

$$\begin{bmatrix} C_1B_1 & C_1B_2 \\ C_2B_1 & C_2B_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -3 \end{bmatrix}$$

which implies that system model (39) has a vector relative degree [1,1]. Then, based on Lemma 1, let $[\xi_1(t), \xi_2(t)]^T = [y_1(t), y_2(t)]^T$ and $\eta(t) = x_3(t)$, and the system model is transformed into two subsystems: the output dynamics

$$\begin{aligned} y_1(t+1) &= -f_1(x(t)) - u_1(t) - u_2(t) \\ y_2(t+1) &= f_1(x(t)) - 2f_2(x(t)) - f_3(x(t)) - 3u_2(t) \end{aligned} \quad (40)$$

and the internal dynamics $\eta(t+1) = f_3(x(t)) + u_1(t)$. Based on the transformed system dynamics, one can verify that the simulation model satisfies Assumptions 1 and 2.

### C. Parameterized Model

With the relative degree condition, the output dynamics (40) are parameterized as $y(t+1) = \Theta_{cf}^*\phi_f(x(t)) + \Theta_{cb}^*u(t)$, where $\phi_f(x) = [x_1, x_1\sin x_3, \sqrt{1+x_2^2}, x_2\sin x_1, x_1, x_3, \sin x_1]^T$, and

$$\Theta_{cf}^* = \begin{bmatrix} -1.2 & -2.08 & -0.24 & 0 & 0 & 0 & 0 \\ 1.2 & 2.08 & 0.24 & -0.32 & -0.8 & -0.48 & -1.6 \end{bmatrix}$$

$$\Theta_{cb}^* = \begin{bmatrix} -1 & -1 \\ 0 & -3 \end{bmatrix}.$$

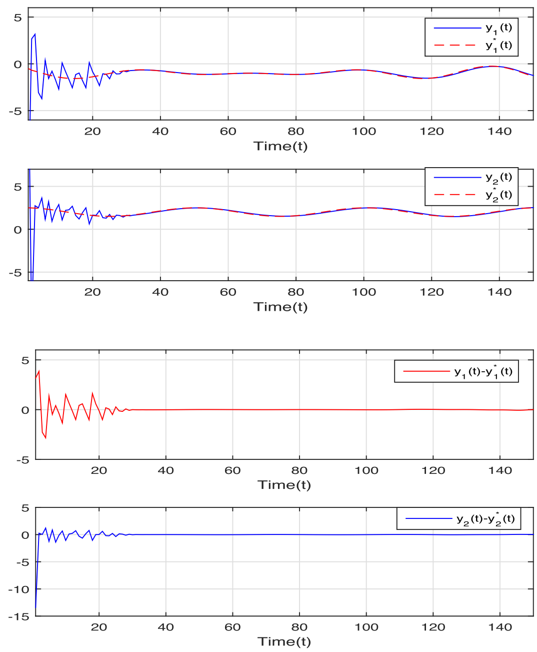


Fig. 1. Response of output $y$ of the model (39) versus $y^*$.

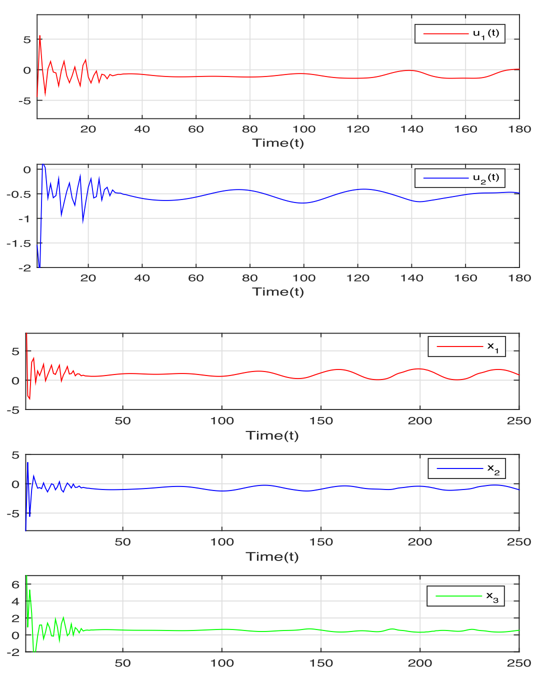


Fig. 2. Response of control input and system state variables.

To derive the parameterized model, we first decompose $\Theta_{cb}^*$ as the SDU form

$$\Theta_{cb}^* = S^* D_s U_s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad (41)$$

where $S^*$, $D_s$, and $U_s$ are the three matrices of the right-hand side of (41), respectively. Then, from (7), the parameterized model of the output dynamics is

$$S^{*-1} y(t+1) = D_s \Theta_1^* \phi_f(x(t)) + D_s \Theta_2^* u(t) + D_s u(t)$$

where

$$S^{*-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \Theta_2^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\Theta_1^* = \begin{bmatrix} 1.2 & 2.08 & 0.24 & 0 & 0 & 0 & 0 \\ -0.4 & -0.69 & -0.08 & 0.1067 & 0.2667 & 0.16 & 0.5333 \end{bmatrix}.$$

### D. Adaptive Control Law

From (8), the adaptive control law is designed as

$$\begin{aligned} u(t) = & -\Theta_2(t)u(t) - \Theta_1(t)\phi_f(x(t)) + \Theta_3(t)y^*(t+1) \\ & -\Theta_3(t)A_m(y(t) - y^*(t)) \end{aligned}$$

where $y^*(t) = [-1 - 0.5\sin\frac{t}{6} + 0.5\cos\frac{t}{7}, 2 + 0.5\cos\frac{t}{8}]^T$, and $\Theta_i(t), i = 1,2,3$, are the estimates of $\Theta_1^*$, $\Theta_2^*$, $(S^* D_s)^{-1}$, respectively. Specifically,

$$\Theta_1(t) = \begin{bmatrix} \theta_{111}(t) & \theta_{112}(t) & \cdots & \theta_{117}(t) \\ \theta_{121}(t) & \theta_{122}(t) & \cdots & \theta_{127}(t) \end{bmatrix}$$

$$\Theta_2(t) = \begin{bmatrix} 0 & \theta_{212}(t) \\ 0 & 0 \end{bmatrix}$$

$$\Theta_3(t) = \begin{bmatrix} \theta_{311}(t) & \theta_{312}(t) \\ \theta_{321}(t) & \theta_{322}(t) \end{bmatrix}$$

and

$$(S^* D_s)^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -0.3333 \end{bmatrix}, \ A_m = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

From (21) and (22), the parameter update laws can be specified. However, to preserve a reasonable article length, we omit further details.

Fig. 1 presents the response of $y(t)$ of the model (39) *versus* $y^*(t)$, which highlights that $y(t)$ tracks $y^*(t)$ asymptotically. Fig. 2 illustrates the response of the control signal $u(t)$ and the system state variables, indicating that the input and the state variables are bounded. In summary, the simulation figures verify the validity of the proposed adaptive control scheme.

## V. Conclusion

This article develops a new matrix decomposition-based solution to solve the singularity problem in an adaptive control design of a class of noncanonical form MIMO DT nonlinear systems with a vector relative degree [1,1,...,1]. The developed control design overcomes the restrictive conditions presented in the literature. The restrictive conditions are the controlled plants of some canonical forms and the high-frequency gain matrices of positive/negative definite. Through simulations, we demonstrate the control design procedure and verify the effectiveness of the proposed control scheme. Further work shall address the adaptive control problem for systems with high-order vector relative degrees and investigate the practical application of the proposed control scheme.

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